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On Super-Mean-Valued Functions and Semi-Polar Sets

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A well-known theorem of Doob—generalizing an important theorem of Cartan in classical potential theory—states that if f is the limit of a decreasing sequence of excessive functions, then f differs from its excessive regularization $\bar{f}$ on a semipolar set. As a matter of fact the proof shows something more in the case when f is bounded: namely that for each $\epsilon > 0$, the set $A_\epsilon = \{f - \bar{f} \geqslant \epsilon\}$ is thin. The purpose of this note is to characterize those super-mean-valued functions f for which the previous statement is true. Our theorem gives a new insight into Doob's theorem. It also yields as an immediate corollary the recent generalization of Doob's theorem given by Getoor and Murali Rao [2].

The notation and terminology used throughout this note are those of [1].

We recall that if $(G, \mathscr{G})$ is a measurable space and $f : G \to \bar{R}$, we write $f \in \mathscr{G}$ if f is measurable with respect to $\mathscr{G}$. We also recall that

$$\mathscr{G}_+ = \{f \in \mathscr{G} \mid f \geqslant 0\} \qquad \text{and} \qquad b\mathscr{G} = \{f \in \mathscr{G} \mid f \text{ bounded}\}.$$

Let $X = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ be a *fixed standard process with state space* $(E, \mathscr{E})$: for the sake of simplicity we assume that $\mathscr{M} = \mathscr{F}$ and $\mathscr{M}_t = \mathscr{F}_t$ for all t. We recall that $\mathscr{E}^*$ denotes the tribe (σ-algebra) of universally measurable subsets of E, $\mathscr{E}^n$ denotes the tribe of nearly Borel subsets of E and that $\mathscr{E} \subset \mathscr{E}^n \subset \mathscr{E}^*$. Finally we recall that a function $f \in \mathscr{E}_+{}^*$ is called α-*super-mean-valued* ($\alpha \geqslant 0$) if $P_t{}^\alpha f \leqslant f$ for all $t \geqslant 0$. If f is α-super-mean-valued then $\bar{f} = \lim_{t \to 0} P_t{}^\alpha f$ exists pointwise, $\bar{f} \leqslant f$, $\bar{f}$ is α-excessive and in fact $\bar{f}$ is the largest α-excessive function dominated by f.[1] Furthermore for each $\beta \geqslant 0$, $U^\beta f = U^\beta \bar{f}$ (see [1, p. 81]). This shows in particular that if $\bar{f}$ is finite then $\{\bar{f} \neq f\}$ is of potential zero.

Let S be a stopping time (all stopping times considered below are taken with respect to $\{\mathscr{F}_t\}$). We shall introduce the following terminology: We

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[1] The function $\bar{f}$ is called the α-excessive regularization of f. Note also that $P_t{}^\alpha f \leqslant \bar{f}$ for all $t > 0$.

shall say that S is a "*delayed hitting time*" if there are $B \in \mathscr{E}^n$ and $t > 0$ such that

$$S = t + T_B \circ \theta_t .$$

We shall need the following more or less known characterization of functions that are upper semi-continuous in the fine topology (since we cannot find it as such in the literature, we shall give the proof below):

PROPOSITION 1. *Let $f \in b\mathscr{E}^n$. Let $\alpha \geqslant 0$. The following assertions are then equivalent:*

(i) *f is upper semi-continuous in the fine topology.*

(ii) *Given any $x \in E$ and (T_n) a sequence of stopping times with $T_n \to 0$ a.s. P^x, we have*

$$\limsup_n P^\alpha_{T_n} f(x) \leqslant f(x).$$

(iii) *Given any $x \in E$ and (T_n) a sequence of delayed hitting times with $T_n \to 0$ a.s. P^x, we have*

$$\limsup_n P^\alpha_{T_n} f(x) \leqslant f(x).$$

Proof. (i) $\Rightarrow$ (ii). Let $x \in E$ and (T_n) a sequence of stopping times with $T_n \to 0$ a.s. P^x. For each $\epsilon > 0$ let $B_\epsilon = \{y \mid f(y) < f(x) + \epsilon\}$. Then $B_\epsilon \in \mathscr{E}^n$, B_ϵ is finely open and $x \in B_\epsilon$. We deduce

$$\limsup_n e^{-\alpha T_n} f(X_{T_n}) \leqslant f(x) + \epsilon \qquad \text{a.s. } P^x;$$

and by Fatou's lemma

$$\limsup_n P^\alpha_{T_n} f(x) \leqslant f(x) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary the implication (i) $\Rightarrow$ (ii) is proved.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $a \in R$ and set $A = \{y \mid f(y) \geqslant a\}$. We must show that A is finely closed, or equivalently that $A^r \subset A$. Hence let $x \in A^r$. Let (K_n) be an increasing sequence of compact sets contained in A, with $T_{K_n} \to 0$ a.s. P^x. Let $t_n > 0$ and set

$$T_n = t_n + T_{K_n} \circ \theta_{t_n} \qquad \text{for each } n;$$

then (T_n) is a sequence of delayed hitting times. We can obviously assume that the t_n's are so chosen that $T_n \to 0$ a.s. P^x. By the Markov property we have

$$X_{T_n} \in K_n \subset A \qquad \text{a.s. on } \{T_n < \infty\},$$

and therefore

$$f(X_{T_n}) \geqslant a \qquad \text{a.s. on } \{T_n < \infty\}.$$

We deduce

$$\liminf_n e^{-\alpha T_n} f(X_{T_n}) \geqslant a \qquad \text{a.s. } P^x,$$

and hence, using Fatou's lemma and the assumption, $f(x) \geqslant a$, i.e. $x \in A$. This completes the proof.

Remark. The implication (iii) $\Rightarrow$ (i) remains valid (with exactly the same proof) for any $f \in \mathscr{E}_+{}^n$.

We can now state our theorem:

THEOREM. *Let $f \in b\mathscr{E}_+{}^n$ be α-super-mean-valued. Let $\bar{f}$ be its α-excessive regularization and for each $\epsilon > 0$ let $A_\epsilon = \{f - \bar{f} \geqslant \epsilon\}$. Then the following assertions are equivalent:*

(i) *A_ϵ is thin for each $\epsilon > 0$.*

(ii) *Almost surely $t \to f(X_t)$ has right hand limits and the right continuous regularization of $t \to f(X_t)$ is $t \to \bar{f}(X_t)$.*

(iii) *Given any $x \in E$ and sequence (T_n) of stopping times with $T_n > 0$ a.s. P^x and $T_n \to 0$ a.s. P^x, we have*

$$\lim_n P^\alpha_{T_n} f(x) \leqslant \bar{f}(x).$$

(iv) *Given any $x \in E$ and sequence (T_n) of delayed hitting times with $T_n \to 0$ a.s. P^x, we have*

$$\limsup_n P^\alpha_{T_n} f(x) \leqslant \bar{f}(x).$$

Proof. (i) $\Rightarrow$ (ii). This is a standard argument. Fix $\epsilon > 0$ and let $T = T_{A_\epsilon}$. Define

$$T_1 = T$$
$$T_{n+1} = T_n + T \circ \theta_{T_n} \qquad \text{for each } n \geqslant 1.$$

Suppose now that β is a countable ordinal and that T_α was defined for all $\alpha < \beta$. If β has a predecessor $\beta - 1$, set

$$T_\beta = T_{\beta-1} + T \circ \theta_{T_{\beta-1}} .$$

If β is a limit ordinal, set

$$T_\beta = \sup_{\alpha<\beta} T_\alpha .$$

In each case T_β is a stopping time. We shall show now that for each countable ordinal β, $T_{\beta+1} > T_\beta$ a.s. on $\{T_\beta < \infty\}$. In fact, since A_ϵ is thin, $T > 0$ almost surely, and we have for each $x \in E$

$$\begin{aligned} P^x\{T_{\beta+1} = T_\beta;\, T_\beta < \infty\} &= P^x\{T \circ \theta_{T_\beta} = 0;\, T_\beta < \infty\} \\ &= E^x\{P^{X(T_\beta)}[T = 0];\, T_\beta < \infty\} = 0. \end{aligned}$$

Hence the assertion is proved.

Fix now $x \in E$. There is then a countable ordinal γ such that

$$P^x(T_\gamma = \infty) = 1.$$

Let $N_x^\epsilon = \{\omega \mid T_\gamma(\omega) < \infty\}$. Then $N_x^\epsilon \in \mathscr{F}$, $P^x(N_x^\epsilon) = 0$ and

$$\begin{aligned} \omega \notin N_x^\epsilon &\Rightarrow \bigcup_{\beta<\gamma} [T_\beta(\omega), T_{\beta+1}(\omega)) = [0, \infty) \\ &\Rightarrow \limsup_{\substack{s\to t \\ s>t}} f(X_s(\omega)) - \bar{f}(X_s(\omega)) \leqslant \epsilon \qquad \text{for each } t \geqslant 0. \end{aligned}$$

Letting ϵ tend to 0 through a sequence we deduce (use the fact that $\bar{f}$ is finely continuous) that $t \to f(X_t(\omega))$ has right hand limits and

$$\lim_{\substack{s\to t \\ s>t}} f(X_s(\omega)) = \bar{f}(X_t(\omega)) \qquad \text{for all } t \geqslant 0,$$

almost surely with respect to P^x. Since x was arbitrary, (i) $\Rightarrow$ (ii) is proved.

Since (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious, it remains only to prove the implication (iv) $\Rightarrow$ (i). Hence assume (iv). By Proposition 1, f is then finely upper semi-continuous and hence so is $f - \bar{f}$. It follows that

$$A_\epsilon^r \subset A_\epsilon \qquad \text{for each } \epsilon > 0.$$

To show that A_ϵ is thin we reason by contradiction. Assume $x \in A_\epsilon^r$. Then $T_{A_\epsilon} = 0$ a.s. P^x. Let

$$T_n = t_n + T_{A_\epsilon} \circ \theta_{t_n}, \qquad \text{where} \qquad t_n > 0 \quad \text{and} \quad t_n \to 0.$$

Then (T_n) is a sequence of delayed hitting times, with $T_n \to 0$ a.s. P^x. By the Markov property we have

$$X_{T_n} \in A_\epsilon \qquad \text{a.s. on } \{T_n < \infty\},$$

whence

$$f(X_{T_n}) - \bar{f}(X_{T_n}) \geqslant \epsilon \qquad \text{a.s. on } \{T_n < \infty\},$$

and thus

$$e^{-\alpha T_n} f(X_{T_n}) \geqslant e^{-\alpha T_n} \bar{f}(X_{T_n}) + \epsilon e^{-\alpha T_n}, \qquad \text{a.s. on } \{T_n < \infty\}.$$

Integrating with respect to P^x we get

$$P^\alpha_{T_n} f(x) \geqslant P^\alpha_{T_n} \bar{f}(x) + \epsilon E^x[e^{-\alpha T_n}; T_n < \infty],$$

and letting $n \to \infty$ we obtain

$$\bar{f}(x) \geqslant \bar{f}(x) + \epsilon,$$

the desired contradiction. This completes the proof of the theorem.

Remark. The implication (iv) $\Rightarrow$ (i) in the above theorem remains valid (same proof) for any α-super-mean-valued function $f \in \mathscr{E}_+{}^n$ for which $\bar{f}$ is finite.

In [2] Getoor and Rao introduced the notion of strongly α-super-mean-valued function. A function f is called *strongly α-super-mean-valued* if:

(a) $f \in \mathscr{E}_+{}^n$;

(b) $P_T{}^\alpha f \leqslant f$ for every stopping time T.

Note that the infimum of two strongly α-super-mean-valued functions is again strongly α-super-mean-valued.

Let us also remark here that if T is a stopping time and t is a strictly positive constant, then (by the Markov property)

$$P^\alpha_{t+T\circ\theta_t} = P_t{}^\alpha P_T{}^\alpha.$$

PROPOSITION 2. *Let f be strongly α-super-mean-valued and let $\bar{f}$ be its α-excessive regularization. We have:*

(1) *If S is a delayed hitting time, then $P_S{}^\alpha f \leqslant \bar{f}$.*

(2) *If S is a terminal time and $S > 0$ a.s., then $P_S{}^\alpha f \leqslant \bar{f}$.*

Proof. Since $f_n = \inf(f, n)$ is strongly α-super-mean-valued and $f_n \uparrow f$, we may assume f bounded and $M = \|f\|_\infty \neq 0$.

Case 1. There are $B \in \mathscr{E}^n$ and $t > 0$ such that $S = t + T_B \circ \theta_t$. We have

$$P_S^\alpha f = P_{t+T_B\circ\theta_t}^\alpha f = P_t^\alpha(P_{T_B}^\alpha f) \leqslant P_t^\alpha f \leqslant \bar{f}.$$

Case 2. Fix $x \in E$ and let $\epsilon > 0$. Since S is a terminal time [1, p. 78], $S = t + S \circ \theta_t$ a.s. on $\{S > t\}$, for each $t \geqslant 0$. As $S > 0$ almost surely, there is then $t > 0$ small enough that

$$P^x(S \neq t + S \circ \theta_t) \leqslant \frac{\epsilon}{M}.$$

We then have:

$$\begin{aligned} P_S^\alpha f(x) &= E^x\{e^{-\alpha S} f(X_S); S \neq t + S \circ \theta_t\} + E^x\{e^{-\alpha S} f(X_S); S = t + S \circ \theta_t\} \\ &\leqslant M \cdot \frac{\epsilon}{M} + E^x\{e^{-\alpha(t+S\circ\theta_t)} f(X_{t+S\circ\theta_t})\} \\ &= \epsilon + P_{t+S\circ\theta_t}^\alpha f(x) = \epsilon + P_t^\alpha(P_S^\alpha f)(x) \\ &\leqslant \epsilon + P_t^\alpha f(x) \leqslant \epsilon + \bar{f}(x). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we obtain $P_S^\alpha f(x) \leqslant \bar{f}(x)$ and the assertion is proved.

COROLLARY (Getoor-Rao; see [2]). *If f is strongly α-super-mean-valued, then $\{\bar{f} < f\}$ is semi-polar.*

Proof. By the usual reduction argument (see [1, p. 81]) we may assume f bounded. The conclusion then follows from Proposition 2 and the preceding Theorem (iv) $\Rightarrow$ (i)).

Remarks. Let f be strongly α-super-mean-valued, let $\bar{f}$ be its α-excessive regularization and assume that $\bar{f}$ is finite. For each $\epsilon > 0$ let $A_\epsilon = \{f - \bar{f} \geqslant \epsilon\}$. Then for each $\epsilon > 0$:

(1) The set A_ϵ is still thin (use Proposition 2 and the Remark at the end of the preceding Theorem).

(2) If $T = T_{A_\epsilon}$, then the following inequality (Getoor-Rao inequality; see [2]) holds for all $x \in E$:

$$\bar{f}(x) \geqslant P_T^\alpha \bar{f}(x) + \epsilon E^x\{e^{-\alpha T}; T < \infty\}.$$

In fact, since $A_\epsilon^r = \varnothing$, we have $X_T \in A_\epsilon$ a.s. on $\{T < \infty\}$. It follows that

$$f(X_T) \geqslant \bar{f}(X_T) + \epsilon \qquad \text{a.s. on } \{T < \infty\}.$$

Multiplying by $e^{-\alpha T}$, integrating with respect to P^x, and noting that $P_T{}^\alpha f \leqslant \hat{f}$ (T is a terminal time and $T > 0$ almost surely, so Proposition 2 applies) yields the desired inequality.

References

1. R. M. Blumenthal and R. K. Getoor, "Markov Processes and Potential Theory," Academic Press, New York and London, 1968.
2. R. K. Getoor and Murali Rao, Another look at Doob's theorem, *Annals of Mathematical Statistics*, 1970, Vol 41, p. 503–506.